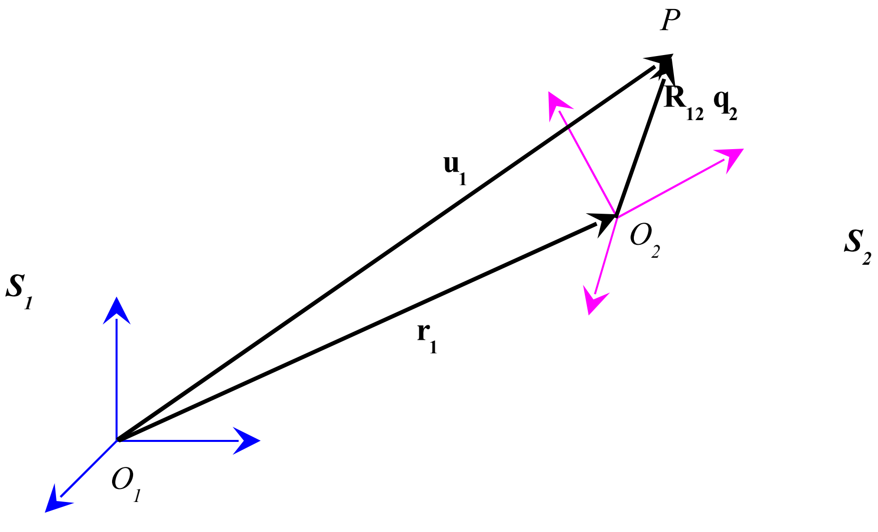
DYNAMICS OF THE RIGID BODY

It's possible to write down the equations for the dynamics of a system composed by material points P_i . Let P_i be fixed reciprocally to form a rigid body. Now the single material point P_i has a mass designated as m_i. In order to study the dynamics of this system, it's necessary to express the acceleration of each point and apply the Newton's laws.
If a rigid body is considered, it's useful to have the mobile reference system $\boldsymbol{S_2}$ fixed with it. In this case, being P_i a point of the rigid body, it's possible to simplify the equations for the velocity and for the acceleration of the point referred to the global reference system $\boldsymbol{S_1}$. This because the vector $\mathbf{q}_2$, expressing the relative position of P_i referred to the center of the mobile reference system $\boldsymbol{S_2}$, is constant. By this it is

$$\mathbf{u}_1 = \mathbf{r}_1 + \mathbf{R}_{12}\mathbf{q}_2 \,, \tag{1}$$

$$\dot{\mathbf{u}}_1 = \dot{\mathbf{r}}_1 + \dot{\mathbf{R}}_{12}\mathbf{q}_2 = \dot{\mathbf{r}}_1 + \boldsymbol{\omega}_1 \times (\mathbf{u}_1 - \mathbf{r}_1) \,, \tag{2}$$

$$\ddot{\mathbf{u}}_1 = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{R}}_{12}\mathbf{q}_2 = \ddot{\mathbf{r}}_1 + \dot{\boldsymbol{\omega}}_1 \times (\mathbf{u}_1 - \mathbf{r}_1) + \boldsymbol{\omega}_1 \times [\boldsymbol{\omega}_1 \times (\mathbf{u}_1 - \mathbf{r}_1)] \,. \tag{3}$$



Momentum and theorem of the momentum

The mathematical relations describing the behavior of a mechanical system are the Newton's law, stating that in a such system the rate of change of momentum equals the resultant of all external forces.
Using the previous formulas we have the following expression for the momentum:

$$\mathbf{p}_1 = \sum_i m_i \dot{\mathbf{u}}_{i,1} = \sum_i m_i \dot{\mathbf{r}}_1 + \sum_i m_i \dot{\mathbf{R}}_{12}\mathbf{q}_{i,2} \,, \tag{4}$$

and using the definition of center of gravity for the given system, it's possible to write down, in the mobile reference system $\boldsymbol{S_2}$,

$$(G - O_2)_2 = \frac{1}{m_{TOT}} \sum_i m_i (P_i - O_2)_2 = \frac{1}{m_{TOT}} \sum_i m_i \mathbf{q_{i,2}} \,. \tag{5}$$

If the origin O_2 of the reference system $\boldsymbol{S_2}$ is coincident with the center of gravity G, it's

$$\mathbf{p_1} = \sum_i m_i \dot{\mathbf{u}}_{\mathbf{i,1}} = \sum_i m_i \dot{\mathbf{r}}_{\mathbf{1}} + \sum_i m_i \dot{\mathbf{R}}_{\mathbf{12}} \mathbf{q_{i,2}} \,, \tag{6}$$

$$\mathbf{p_1} = m_{TOT} \left[\dot{\mathbf{r}}_{\mathbf{1}} + \boldsymbol{\omega}_{\mathbf{1}} \times \mathbf{R_{12}} (G - O_2)_2 \right], \tag{7}$$

finally the momentum is

$$\mathbf{p_1} = \sum_i m_i \dot{\mathbf{u}}_{\mathbf{i,1}} = m_{TOT} \mathbf{v_{G,1}} \,. \tag{8}$$

In fact, the member within squared parenthesis in equation (7) is the velocity of the center of gravity of the system. In accordance with the statement of Newton's law, we have in the reference systems $\boldsymbol{S_1}$ and $\boldsymbol{S_2}$

$$\dot{\mathbf{p}}_{\mathbf{1}} = m_{TOT} \dot{\mathbf{v}}_{\mathbf{G,1}} = \sum_i \mathbf{F_{i,1}}^{\mathbf{ext}} \,, \tag{9}$$

$$\dot{\mathbf{p}}_{\mathbf{2}} = m_{TOT} \dot{\mathbf{v}}_{\mathbf{G,2}} = \sum_{\mathbf{i}} \mathbf{F_{i,2}}^{\mathbf{ext}} \,. \tag{10}$$

Angular momentum

To define the moment of momentum or the angular momentum, a reference point has to be selected. Let O_2 be this point, so the angular momentum referred to O_2 is

$$\mathbf{L}_1(O_2) = \sum_i (P_i - O_2) \times m_i \dot{\mathbf{u}}_{\mathbf{i},1} . \tag{11}$$

Different steps of calculation will be shown, in the following section, to demonstrate as the final and well known expression of the angular momentum comes out. Using the previous equations it's possible to obtain:

$$\mathbf{L}_1(O_2) = \sum_i \mathbf{R}_{12}\mathbf{q}_{\mathbf{i},2} \times m_i \dot{\mathbf{u}}_{\mathbf{i}} ,$$

$$\mathbf{L}_1(O_2) = \sum_i \mathbf{R}_{12}\mathbf{q}_{\mathbf{i},2} \times m_i \left(\dot{\mathbf{r}} + \dot{\mathbf{R}}_{12}\mathbf{q}_{\mathbf{i},2}\right) ,$$

$$\mathbf{L}_1(O_2) = \sum_i \mathbf{R}_{12}\mathbf{q}_{\mathbf{i},2} \times m_i \left(\dot{\mathbf{r}} + \mathbf{R}_{12}\left(\boldsymbol{\omega}_2 \times \mathbf{q}_{\mathbf{i},2}\right)\right) ,$$

$$\mathbf{L}_1(O_2) = \sum_i m_i \mathbf{R}_{12}\mathbf{q}_{\mathbf{i},2} \times \dot{\mathbf{r}} + \sum_i m_i \mathbf{R}_{12}\mathbf{q}_{\mathbf{i},2} \times \mathbf{R}_{12}\left(\boldsymbol{\omega}_2 \times \mathbf{q}_{\mathbf{i},2}\right) ,$$

$$\mathbf{L}_1(O_2) = \sum_i m_i (P - O_2) \times \mathbf{v}_{\mathbf{O}_2,1} + \mathbf{R}_{12}\sum_i m_i \mathbf{q}_{\mathbf{i},2} \times \left(\boldsymbol{\omega}_2 \times \mathbf{q}_{\mathbf{i},2}\right) . \tag{12}$$

Using equation (5) and the definition of the tensor of inertia for a discrete material system, it's possible to obtain the final formula of the angular momentum as

$$\mathbf{L}_1(O_2) = m_{TOT}(G - O_2) \times \mathbf{v}_{\mathbf{O}_2,1} + \mathbf{R}_{12}\mathbf{J}_2\boldsymbol{\omega}_2 ,$$

$$\mathbf{L}_1(O_2) = m_{TOT}(G - O_2) \times \mathbf{v}_{\mathbf{O}_2,1} + \mathbf{R}_{12}\mathbf{J}_2\mathbf{R}_{12}^{\mathbf{T}}\boldsymbol{\omega}_1 ,$$

$$\mathbf{L}_1(O_2) = m_{TOT}(G - O_2) \times \mathbf{v}_{\mathbf{O}_2,1} + \mathbf{J}_1\boldsymbol{\omega}_1 . \tag{13}$$

In the equation (13), $\mathbf{L}_1$ is the angular momentum vector defined in the global reference system, $\mathbf{J}_1$ is the tensor of inertia of our system defined in the global reference system too and, for this reason, its value is not constant. To have a constant tensor of inertia we must use the vector base of the mobile reference system fixed with the body. Let $\mathbf{J}_2$ be the tensor in the mobile reference system and let $\boldsymbol{\omega}_2 = \mathbf{R}_{12}^{-1}\boldsymbol{\omega}_1 = \mathbf{R}_{12}^{\mathbf{T}}\boldsymbol{\omega}_1$ be the vector of the angular velocity in the mobile reference system. Therefore, by calculating the angular momentum referred to the center of gravity G of the body, the first term of the second member of the equation (13) is zero. In fact

$$\mathbf{L}_1(G) = \mathbf{J}_1(G)\boldsymbol{\omega}_1 = \mathbf{R}_{12}\mathbf{J}_2(G)\mathbf{R}_{12}^{\mathrm{T}}\boldsymbol{\omega}_1 = \mathbf{R}_{12}\mathbf{J}_2(G)\boldsymbol{\omega}_2 = \mathbf{R}_{12}\mathbf{L}_2(G), \quad (14)$$

$$\mathbf{L}_2(G) = \mathbf{J}_2(G)\boldsymbol{\omega}_2\,, \qquad (15)$$

where $\mathbf{L}_2$ is the angular momentum in the mobile reference system.
Using the mobile reference system, it's possible to have a more simple formulation for the equations of motion, with constant tensors of inertia.

THEOREM OF THE ANGULAR MOMENTUM

The Newton's law for moments states that in a mechanical system the rate of change of angular momentum equals the resultant of all external moments:

$$\dot{\mathbf{L}}_1(G) = \mathbf{J}_1(G)\dot{\boldsymbol{\omega}}_1 + \boldsymbol{\omega}_1 \times \mathbf{J}_1(G)\boldsymbol{\omega}_1 = \sum_i \mathbf{C}_{\mathrm{i,1}}^{\mathrm{ext}}\,, \qquad (16)$$

where $\mathbf{C}_{\mathrm{i,1}}^{\mathrm{ext}}$ are the external moments applied to different points P_i. To write the equation (16) in the mobile reference system we can use the rotation matrix as done before:

$$\mathbf{R}_{12}\dot{\mathbf{L}}_2(G) = \left(\mathbf{R}_{12}\mathbf{J}_2(G)\mathbf{R}_{12}^{\mathrm{T}}\right)\mathbf{R}_{12}\dot{\boldsymbol{\omega}}_2 + \mathbf{R}_{12}\boldsymbol{\omega}_2 \times \left(\mathbf{R}_{12}\mathbf{J}_2(G)\mathbf{R}_{12}^{\mathrm{T}}\right)\mathbf{R}_{12}\boldsymbol{\omega}_2\,,$$

$$\mathbf{R}_{12}\dot{\mathbf{L}}_2(G) = \mathbf{R}_{12}\mathbf{J}_2(G)\dot{\boldsymbol{\omega}}_2 + \mathbf{R}_{12}\boldsymbol{\omega}_2 \times \mathbf{R}_{12}\mathbf{J}_2(G)\boldsymbol{\omega}_2\,,$$

$$\mathbf{R}_{12}\dot{\mathbf{L}}_2(G) = \mathbf{R}_{12}\mathbf{J}_2(G)\dot{\boldsymbol{\omega}}_2 + \mathbf{R}_{12}\left(\boldsymbol{\omega}_2 \times \mathbf{J}_2(G)\boldsymbol{\omega}_2\right).$$

Finally the time derivative of angular momentum is obtained in the mobile reference system where the tensor of inertia is independent of the time.

$$\dot{\mathbf{L}}_2(G) = \mathbf{J}_2(G)\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{J}_2(G)\boldsymbol{\omega}_2\,. \qquad (17)$$

Therefore the theorem in the mobile reference system is

$$\mathbf{J}_2(G)\dot{\boldsymbol{\omega}}_2 + \boldsymbol{\omega}_2 \times \mathbf{J}_2(G)\boldsymbol{\omega}_2 = \sum_i \mathbf{C}_{\mathrm{i,2}}^{\mathrm{ext}}\,. \qquad (18)$$