DYNAMICS OF MULTI-ROTOR SYSTEMS

The aim of this document is to describe mathematically the behavior of a mechanical system composed by a stator with a different number of rotors. Each body is considered as rigid and their equilibrium equations are similar.
In this section, some mathematical concepts as the shift matrix will be introduced. We will explain as forces can be shifted from a point to another by using this important matrix. Furthermore, we will analyze forces between rotor and stator exchanged by the bearings, and forces among rotors generally due to the transmission mechanism. As an example, in the following calculus a gear transmission will be studied. The same analysis is valid for any other type of transmission.
In order to introduce the dynamics of a multi-rotor system, a system with one stator an two rotors will be analyzed before to generalize the equations.
It's important to note that the stator is supported by a system of suspensions with complex stiffness and damping properties. The suspensions allow the system to produce only small rotations. This condition allows one to simplify the equations of motion and the calculus of the transformation matrix for the experimental analysis.
The important concepts of gyroscopic effect, gyroscopic matrix and stereo-nodal reference system will be introduced.

SHIFT OF GENERALIZED FORCES

A generalized force is a very useful method to write down a six components vector considering forces and moments all together. In fact, the first three components are the forces in x, y, z direction and the second three are the moments in a given point of the mechanical system. Obviously, a generalized force is referred to a point P and to shift this force to another point it's possible to use the shift matrix. This matrix has very important properties that will result useful to simplify the equations of the dynamics. The shift matrix will be used also to express the acceleration of a point with reference to another.
To introduce the shift matrix, let's consider the vector product $\mathbf{b} = (P - O) \times \mathbf{a}$ as

$$\begin{vmatrix} b_x \\ b_y \\ b_z \end{vmatrix} = \begin{vmatrix} 0 & -(z_P - z_O) & +(y_P - y_O) \\ +(z_P - z_O) & 0 & -(x_P - x_O) \\ -(y_P - y_O) & +(x_P - x_O) & 0 \end{vmatrix} \begin{vmatrix} a_x \\ a_y \\ a_z \end{vmatrix}, \tag{1}$$

let A be the matrix that allows to calculate this vector product as $\mathbf{b} = \mathbf{A_{PO}} \times \mathbf{a}$,

$$\mathbf{A_{PO}} = \begin{vmatrix} 0 & -(z_P - z_O) & +(y_P - y_O) \\ +(z_P - z_O) & 0 & -(x_P - x_O) \\ -(y_P - y_O) & +(x_P - x_O) & 0 \end{vmatrix}, \tag{2}$$

where $(P - O)$ is the vector that gives the distance between the two points P and O.

The matrix $\mathbf{A_{PO}}$ is a skew symmetric matrix and, for this reason, we have the following properties:

$$\mathbf{A_{PO}}^{\mathbf{T}} = -\mathbf{A_{PO}} = \mathbf{A_{OP}} \,. \tag{3}$$

Let's define the vector of generalized forces, with reference to the point *O*, as

$$\mathbf{T_O} = \begin{vmatrix} F_x \\ F_y \\ F_z \\ C_x \\ C_y \\ C_z \end{vmatrix}_O \,. \tag{4}$$

If we want to shift this forces and moments from *O* to another point *P*, the first three components that gives the forces in the three directions remain the same, but moments have to be calculate with reference to the new point *P*. For this reason we can write

$$\begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}_P = \begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}_O \,, \tag{5}$$

$$\begin{vmatrix} C_x \\ C_y \\ C_z \end{vmatrix}_P = \begin{vmatrix} C_x \\ C_y \\ C_z \end{vmatrix}_O + (O - P) \times \begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}_O = \begin{vmatrix} C_x \\ C_y \\ C_z \end{vmatrix}_O - (P - O) \times \begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}_O = \begin{vmatrix} C_x \\ C_y \\ C_z \end{vmatrix}_O - \mathbf{A_{PO}} \times \begin{vmatrix} F_x \\ F_y \\ F_z \end{vmatrix}_O \,. \tag{6}$$

Now, it's possible to express the generalized force in the point *P* as

$$\begin{vmatrix} F_x \\ F_y \\ F_z \\ C_x \\ C_y \\ C_z \end{vmatrix}_P = \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{A_{PO}} & \mathbf{I} \end{vmatrix} \begin{vmatrix} F_x \\ F_y \\ F_z \\ C_x \\ C_y \\ C_z \end{vmatrix}_O \,. \tag{7}$$

Finally it's possible to define the shift matrix as

$$\mathbf{B}_{\mathbf{PO}} = \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ -\mathbf{A}_{\mathbf{PO}} & \mathbf{I} \end{vmatrix}, \tag{8}$$

that gives the possibility to write the equation (7) in the following compact form:

$$\mathbf{T}_{\mathbf{P}} = \mathbf{B}_{\mathbf{PO}}\mathbf{T}_{\mathbf{O}} . \tag{9}$$

A remarkable property of this matrix regards successive products. In fact, let *O*, *P* and *N* three points of a system, if we want to shift a force consequently from *O* to *P* and to *N*, it's

$$\mathbf{T}_{\mathbf{P}} = \mathbf{B}_{\mathbf{PO}}\mathbf{T}_{\mathbf{O}} ,$$

$$\mathbf{T}_{\mathbf{N}} = \mathbf{B}_{\mathbf{NP}}\mathbf{T}_{\mathbf{P}} ,$$

$$\mathbf{T}_{\mathbf{N}} = \mathbf{B}_{\mathbf{NP}}\mathbf{B}_{\mathbf{PO}}\mathbf{T}_{\mathbf{O}} = \mathbf{B}_{\mathbf{NO}}\mathbf{T}_{\mathbf{O}} .$$

Therefore we have the following properties:

$$\mathbf{B}_{\mathbf{NO}} = \mathbf{B}_{\mathbf{NP}}\mathbf{B}_{\mathbf{PO}} , \tag{10}$$

$$\mathbf{B}_{\mathbf{PO}}^{-1} = \mathbf{B}_{\mathbf{OP}} . \tag{11}$$

ROTOR-STATOR INTERACTION FORCES

The forces between rotor and stator are of different types:

- forces exchanged in the bearings;
- torques due to the magnetic interaction between rotor and stator in the electric motor;
- forces due to interaction between driven rotor and stator as friction forces;
- forces due to the transmission mechanism: gears or driven belts.

Forces in the bearings

Being the bodies considered as rigid, the effect of the two bearings is condensed in a single point acting as a joint while the rotor acts as a rigid cantilever beam. Generally, the point considered to express the generalized force is the position of one bearing. This force will be calculated by the shift matrix in the center of gravity of the stator chosen as origin for the stereo-nodal reference system. The generalized force, expressing the resultant forces and moments for the bearings of the rotor j-th, is defined as

$$\mathbf{T}_{\mathbf{Bj}} = \begin{vmatrix} \mathbf{F}_{\mathbf{Bj}} \\ \mathbf{C}_{\mathbf{Bj}} \end{vmatrix} . \qquad (12)$$

If *S* is the center of gravity of the stator, we can calculate the forces of the bearings referred to the point *S* by using the shift matrix:

$$\mathbf{T}_{\mathbf{S}} = \mathbf{B}_{\mathbf{SBj}} \mathbf{T}_{\mathbf{Bj}} . \qquad (13)$$

Forces due to the transmission

Let's consider the common case of an electric motor and a driven rotor. Let's suppose to have a variable system of external forces acting on the rotor. We can recognize the resistant torque, the magnetic torque, the centrifugal forces due to unbalanced masses and the friction forces.
It's important to analyze deeply the presence of this forces and moments in the equilibrium, because often, studying the transmission mechanism of a static system, the magnetic torque acting on the stator, by means of the electric action of the motor, are not considered because they act on a rigid basement fixed to the ground. Instead, in our case these forces are fundamental in order to write down the correct equilibrium equation. In this section a gear transmission will be analyzed, but it's possible to make the same analysis for any type of transmission mechanism.

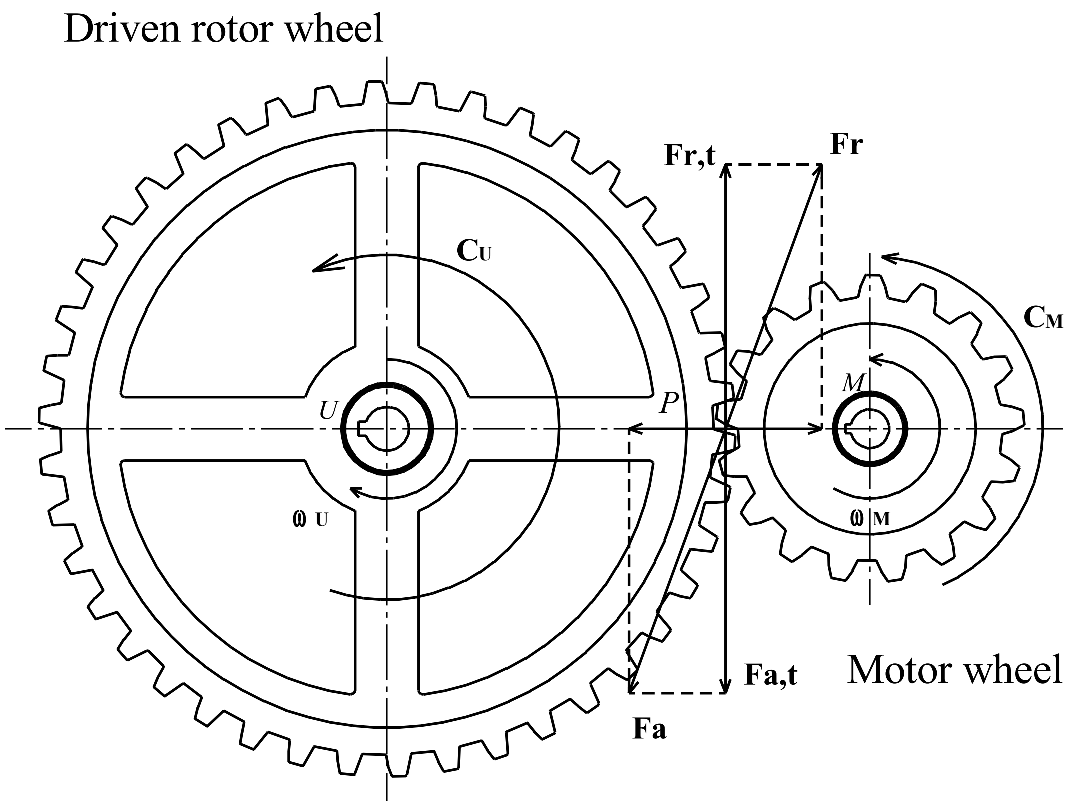

Figure 2

Let’s consider the gear in figure 2 to transmit the motion from the motor to the driven rotor, let τ be the transmission ratio defined as

$$\tau = \frac{|\omega_M|}{|\omega_U|} = \frac{R_U}{R_M}, \tag{14}$$

where ω_M and ω_U are the angular velocities of the two gears and R_U and R_M are the mean radiuses of the tooth wheels.
By using the shift matrix, it’s possible to shift the active force $\mathbf{F_a}$ and the reactive force $\mathbf{F_r}$ from the point *P* to the center of the driven rotor called *U* and to the center of the motor called *M* obtaining the common results

$$\mathbf{F_U} = \mathbf{B_{UP}} \mathbf{F_a}, \tag{15}$$

$$\mathbf{F_M} = \mathbf{B_{MP}} \mathbf{F_r}. \tag{16}$$

By considering the value of the angular velocities with their orientation, it's possible to write a relation between the two angular velocities, but also between the two angular accelerations of the two rotors. In fact

$$\omega_U = -\frac{\omega_M}{\tau}, \tag{17}$$

$$\alpha_U = -\frac{\alpha_M}{\tau}. \tag{18}$$

The reason of the negative sign is in the rotation versus of the two tooth wheels in the gear transmission. If the transmission was obtained by belt and pulley, the signs would be positive. Now, the equilibrium of the forces acting onto the system, has to be expressed by the usual laws for the motor shaft and for the driven rotor. The two equations are

$$C_U - F_{a,t} R_U = J_U \alpha_U, \tag{19}$$

$$C_M - F_{r,t} R_M = J_M \alpha_M, \tag{20}$$

where, as said before, R_U and R_M are respectively the means radiuses of the tooth wheel of the driven rotor and the radius of the tooth wheel of the motor; C_U is the resistant torque and C_M is the motor torque; $F_{a,t}$ and $F_{r,t}$ are the tangential action and reaction forces. The nature of C_U could be very different: unbalances, fluid dynamic resistance, friction etc., C_M, in the common case of an electric motor is a magnetic torque received directly by the stator too.
But for the Newton's law it's also

$$F_{a,t} = F_{r,t} \tag{21}$$

and using equations (18), (19) and (20) we obtain the following important relations for the equilibrium of a transmission mechanism:

$$F_{a,t} = \frac{C_U + J_U \alpha_U}{R_U}, \tag{22}$$

$$C_M = \frac{C_U}{\tau} + \left(\frac{J_U}{\tau^2} + J_M \right) \alpha_M. \tag{23}$$

In the case of a transmission with belt and pulley it's simple to find $C_M = \frac{C_U}{\tau} - \left(\frac{J_U}{\tau^2} + J_M \right) \alpha_M$.

Forces in the suspensions

The forces exchanged between the stator and the suspensions depends on their mathematical formulation. For the moment this forces will be considered as generalized forces without to show explicitly their constitutive equations. In the numerical analysis, we will choose a mathematical model for springs and dampers, in order to reproduce as much as possible the experimental results. It will be explained as their formulation is at same time difficult and fundamental in order to reproduce the true behavior of the system.
The proof the numerical model for the suspensions is correct will come directly from the comparison between experimental and numerical results.

Equations for a stator with two rotors

In this analysis, the stator is a rigid body interested by the forces described in the previous sections. The rotors are also considered as rigid bodies with parallel axes of rotation. This is a very common configuration, and the hypothesis of rigid body is correct when the deformation of the structure is negligible compared with its displacements in the space.
A stator with n rotors is a simple generalization of a system composed by a stator with two rotors. If we consider a system like this, four reference frames are necessary. One reference system will be the global reference $\boldsymbol{S_1}$ fixed to the ground, another $\boldsymbol{S_2}$ will be fixed to the stator and the other two $\boldsymbol{S_3}$ and $\boldsymbol{S_4}$ will be fixed to the rotors. Generally the reference system $\boldsymbol{S_2}$ fixed to the stator – the stereo-nodal reference system – won't be a principal system, i.e., the moments of inertia of the stator in this system won't be principal. The reason is that an axis of this reference frame must be parallel to the rotation axis of the rotors to solve the problem easily.
Instead, the reference systems $\boldsymbol{S_3}$ and $\boldsymbol{S_4}$ will be principal for the rotors and the tensor of inertia will be principal with a gyroscopic structure, that means a polar or axial moment of inertia and two equal diametral moments of inertia. This is the most important property of this system because the tensor of inertia of the rotors doesn't change if they are spinning respect to a reference frame fixed to the stator. At this point the only thing to consider is the effect of rotation and this will be done by the gyroscopic matrix. For simplicity, in figure 3 only three reference systems are shown, the global, the reference fixed to the stator and the reference fixed to the first rotor.
Point O_1 is the center of the global reference, P_S is the center of gravity of the stator and P_R is the center of gravity of the first rotor.

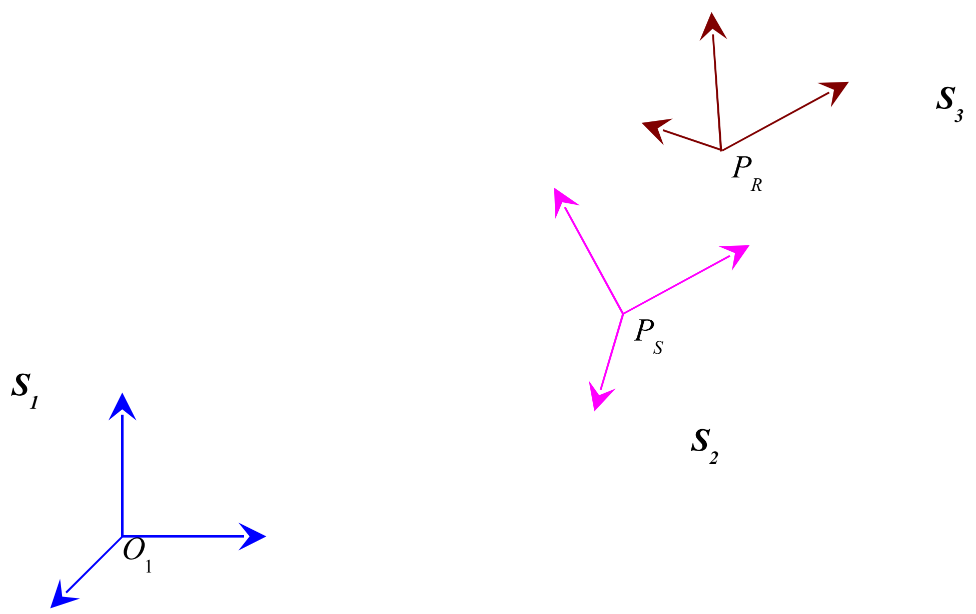

Figure 3

Dynamics of the stator

As said before, the equations for the dynamics of the different bodies will be written in the mobile reference systems fixed to them where the tensor of inertia is independent of the time.
It’s immediately possible to find out the equation for the stator in its reference system by using the equations (3.15) and (3.17):

$$\mathbf{L}_{\mathbf{S},2}(P_S) = \mathbf{J}_{\mathbf{S},2}\boldsymbol{\omega}_{\tau,2}\,, \tag{24}$$

$$\dot{\mathbf{L}}_{\mathbf{S},2}(P_S) = \mathbf{J}_{\mathbf{S},2}(P_S)\dot{\boldsymbol{\omega}}_{\tau,2} + \boldsymbol{\omega}_{\tau,2} \times \mathbf{J}_{\mathbf{S},2}(P_S)\boldsymbol{\omega}_{\tau,2} = \sum_i \mathbf{C}_{\mathbf{S},\mathbf{i},2}\,. \tag{25}$$

$\boldsymbol{\omega}_{\tau,2}$ is the drag angular velocity defined by the motion of the reference $\boldsymbol{S_2}$ referred to $\boldsymbol{S_1}$ and it’s written in its vector base. $\mathbf{J}_{\mathbf{S},2}$ is the tensor of inertia of the stator and generally is not principal as already said.
To describe completely the equilibrium of this body the equation for the force equilibrium is needed. We can put together this equation and the equation (25) in order to obtain the following system of vector equations:

$$\begin{cases} m_S\ddot{\mathbf{u}}_{\mathbf{S},2} = \sum_{\mathbf{i}} \mathbf{F}_{\mathbf{S},\mathbf{i},2} \\ \\ \mathbf{J}_{\mathbf{S},2}\dot{\boldsymbol{\omega}}_{\tau,2} + \boldsymbol{\omega}_{\tau,2} \times \mathbf{J}_{\mathbf{S},2}\boldsymbol{\omega}_{\tau,2} = \sum_{\mathbf{i}} \mathbf{C}_{\mathbf{S},\mathbf{i},2} \end{cases}. \tag{26}$$

In the second member of this vector equations all the forces are condensed in the summation.
Now it’s possible to obtain a more simple formulation using a single vector equation. It’s important to take into account that $\sum_{\mathbf{i}} \mathbf{C}_{\mathbf{S},\mathbf{i},2}$ is the summation of pure moments and moments caused by forces applied in the points P_i of the stator and calculated with reference to the vector base of $\boldsymbol{S_2}$. If we call $\mathbf{N}_{\mathbf{S},\mathbf{i}}$ the pure moments in the points P_i of the stator in $\boldsymbol{S_2}$, using the shift matrix to calculate the forces in the center of gravity indicated with S, we obtain

$$\begin{vmatrix} m_S\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_\mathbf{S} \end{vmatrix}\begin{vmatrix} \ddot{\mathbf{u}}_\mathbf{S} \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \begin{vmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Gamma}\mathbf{J}_\mathbf{S} \end{vmatrix}\begin{vmatrix} \dot{\mathbf{u}}_\mathbf{S} \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_{\mathbf{i}} \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{A}_{\mathbf{i},\mathbf{S}} & \mathbf{I} \end{vmatrix}\begin{vmatrix} \mathbf{F}_{\mathbf{S},\mathbf{i}} \\ \mathbf{N}_{\mathbf{S},\mathbf{i}} \end{vmatrix}. \tag{27}$$

In this equation we used the matrix $\boldsymbol{\Gamma}$ that defines the angular velocity of the system $\boldsymbol{S_2}$ referred to the system $\boldsymbol{S_1}$. From now, we will make our analysis in the reference frame $\boldsymbol{S_2}$ only. Therefore we will neglect the pedice 2 in the quantities of the next equations.

By using the matrix product to express the second quantity of the equation (27), it's possible to write

$$\begin{vmatrix} m_S\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_S \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \begin{vmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Gamma} \end{vmatrix} \begin{vmatrix} m_S\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_S \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_{\mathbf{i}} \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{A}_{i,S} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{F}_{S,i} \\ \mathbf{N}_{S,i} \end{vmatrix}. \tag{28}$$

Let's define the global mass matrix $\mathbf{M}_S$ and the generalized vector of forces and moments $\mathbf{T}_{S,i}$ as

$$\mathbf{M}_S = \begin{vmatrix} m_S\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_S \end{vmatrix}, \tag{29}$$

$$\mathbf{T}_{S,i} = \begin{vmatrix} \mathbf{F}_{S,i} \\ \mathbf{N}_{S,i} \end{vmatrix}. \tag{30}$$

Using these definitions for the vectors, it's possible to obtain a more condensed vector equation useful for the next analysis:

$$\mathbf{M}_S \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_S \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_{\mathbf{i}} \mathbf{B}_{S,i}\mathbf{T}_{S,i} \ . \tag{31}$$

In this last equation we introduced the 6 by 6 matrix $\boldsymbol{\Lambda}$ that contains the 3 by 3 matrix $\boldsymbol{\Gamma}$ of the angular velocity seen in equation (2.5).

Dynamics of the rotor

For the rotor, as already done for the stator, it's possible to write the angular momentum referred to its center of gravity *R* in the three reference systems as

$$\mathbf{L}_{\mathbf{R},3}(R) = \mathbf{J}_{\mathbf{R},3}\boldsymbol{\omega}_{\mathbf{a},3}\,, \tag{32}$$

$$\mathbf{L}_{\mathbf{R},2}(R) = \mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}(R)\,, \tag{33}$$

$$\mathbf{L}_{\mathbf{R},1}(R) = \mathbf{R}_{12}\left(\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}(R)\right). \tag{34}$$

In this three equations, the quantity is always the angular momentum of the rotor written with reference to three reference frames by using the rotation matrixes. By deriving with respect to time the equation (34) we obtain the following steps:

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \frac{d}{dt}\left(\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right), \tag{35}$$

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \dot{\mathbf{R}}_{12}\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3} + \mathbf{R}_{12}\frac{d}{dt}\left(\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right), \tag{36}$$

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\left(\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right) + \mathbf{R}_{12}\frac{d}{dt}\left(\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right), \tag{37}$$

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \boldsymbol{\omega}_{\tau,1} \times \left(\mathbf{R}_{12}\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right) + \mathbf{R}_{12}\frac{d}{dt}\left(\mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3}\right), \tag{38}$$

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \mathbf{R}_{12}\boldsymbol{\omega}_{\tau,2} \times \left(\mathbf{R}_{12}\mathbf{L}_{\mathbf{R},2}\right) + \mathbf{R}_{12}\dot{\mathbf{L}}_{\mathbf{R},2}\,, \tag{39}$$

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \mathbf{R}_{12}\left(\boldsymbol{\omega}_{\tau,2} \times \mathbf{L}_{\mathbf{R},2}\right) + \mathbf{R}_{12}\dot{\mathbf{L}}_{\mathbf{R},2}\,. \tag{40}$$

From the equation (40) we will obtain the final moment equilibrium for the rotor. Let's evaluate the angular momentum $\mathbf{L}_{\mathbf{R},2}$:

$$\mathbf{L}_{\mathbf{R},2} = \mathbf{R}_{23}\mathbf{L}_{\mathbf{R},3} = \mathbf{R}_{23}\left(\mathbf{J}_{\mathbf{R},3}\boldsymbol{\omega}_{\mathbf{a},3}\right) = \mathbf{R}_{23}\mathbf{J}_{\mathbf{R},3}\mathbf{R}_{23}^{\mathbf{T}}\boldsymbol{\omega}_{\mathbf{a},2} = \mathbf{J}_{\mathbf{R},2}\boldsymbol{\omega}_{\mathbf{a},2} \tag{41}$$

By some considerations on the characteristics of matrix $\mathbf{R}_{23}$ and of the vector $\mathbf{L}_{\mathbf{R},3}$, it's possible to obtain an important result for the theory of the dynamics of multi-rotor systems.

Matrix $\mathbf{R}_{23}$ is the rotation matrix that transform vectors from the base of the mobile system $\boldsymbol{S_3}$ to the base of the mobile system $\boldsymbol{S_2}$. System $\boldsymbol{S_3}$ is referred to the rotor and $\boldsymbol{S_2}$ to the stator. We said reference system $\boldsymbol{S_2}$ has been chosen in order to place one of its axis parallel to the same axis of $\boldsymbol{S_3}$ chosen as rotation axis.
If x is the rotation axis of the rotor, the matrix $\mathbf{R}_{23}$ will have the following form:

$$\mathbf{R}_{23} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_R t) & -\sin(\omega_R t) \\ 0 & \sin(\omega_R t) & \cos(\omega_R t) \end{vmatrix}, \tag{42}$$

where ω_R is the angular spinning speed of the rotor respect to the stator and therefore $\omega_R t$ is the angle between reference $\boldsymbol{S_2}$ and $\boldsymbol{S_3}$.
The vector $\mathbf{L}_{\mathbf{R},3}$ is the angular momentum of the rotor expressed in the vector base of the reference system $\boldsymbol{S_3}$ fixed to it, and $\mathbf{J}_{\mathbf{R},3}$ is the tensor if inertia of the rotor that, for its gyroscopic structure, has the following shape;

$$\mathbf{J}_{\mathbf{R},3} = \begin{vmatrix} J_p & 0 & 0 \\ 0 & J_d & 0 \\ 0 & 0 & J_d \end{vmatrix}. \tag{43}$$

If the gyroscopic tensor of inertia of the rotor, defined by means of the vector base of the reference system $\boldsymbol{S_3}$, is transformed in the system $\boldsymbol{S_2}$, for the properties of the rotation matrix we have

$$\mathbf{J}_{\mathbf{R},2} = \mathbf{R}_{23}\mathbf{J}_{\mathbf{R},3}\mathbf{R}_{23}^{\mathbf{T}} = \mathbf{J}_{\mathbf{R}}, \tag{44}$$

in fact

$$\mathbf{J}_{\mathbf{R}} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_R t) & -\sin(\omega_R t) \\ 0 & \sin(\omega_R t) & \cos(\omega_R t) \end{vmatrix} \begin{vmatrix} J_p & 0 & 0 \\ 0 & J_d & 0 \\ 0 & 0 & J_d \end{vmatrix} \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\omega_R t) & \sin(\omega_R t) \\ 0 & -\sin(\omega_R t) & \cos(\omega_R t) \end{vmatrix} = \begin{vmatrix} J_p & 0 & 0 \\ 0 & J_d & 0 \\ 0 & 0 & J_d \end{vmatrix}. \tag{45}$$

That means the inertial tensor of the rotor in the stereo-nodal reference system $\boldsymbol{S_2}$ has the same form as in the reference $\boldsymbol{S_3}$, but the most important result is the tensor of inertia in $\boldsymbol{S_2}$ is independent of the time. Because this, equation (40) becomes

$$\frac{d}{dt}\left(\mathbf{L}_{\mathbf{R},1}\right) = \mathbf{R}_{12}\left(\boldsymbol{\omega}_{\boldsymbol{\tau},2} \times \mathbf{J}_{\mathbf{R}}\boldsymbol{\omega}_{\mathbf{a},2} + \mathbf{J}_{\mathbf{R}}\dot{\boldsymbol{\omega}}_{\mathbf{a},2}\right) \tag{46}$$

and by using the relation $\boldsymbol{\omega}_{a,2} = \boldsymbol{\omega}_{\tau,2} + \boldsymbol{\omega}_{r,2}$ and applying the theorem of the angular momentum in the stereo-nodal reference system, we obtain the final equation for the moment equilibrium of the rotor:

$$\boldsymbol{\omega}_{\tau,2} \times \mathbf{J}_{\mathbf{R}}\left(\boldsymbol{\omega}_{\tau,2} + \boldsymbol{\omega}_{r,2}\right) + \mathbf{J}_{\mathbf{R}}\left(\dot{\boldsymbol{\omega}}_{\tau,2} + \dot{\boldsymbol{\omega}}_{r,2}\right) = \sum_{\mathbf{i}} \mathbf{C}_{\mathbf{R,i,2}} \,. \tag{47}$$

To describe completely the equilibrium of this body, as already done for the stator, the equation for the force equilibrium is needed. With this equation we can write down the following system of vector equations for the rotor:

$$\begin{cases} m_R \ddot{\mathbf{u}}_{\mathbf{R,2}} = \sum_{\mathbf{i}} \mathbf{F}_{\mathbf{R,i,2}} \\ \mathbf{J}_{\mathbf{R}}\left(\dot{\boldsymbol{\omega}}_{\tau,2} + \dot{\boldsymbol{\omega}}_{r,2}\right) + \boldsymbol{\omega}_{\tau,2} \times \mathbf{J}_{\mathbf{R}}\left(\boldsymbol{\omega}_{\tau,2} + \boldsymbol{\omega}_{r,2}\right) = \sum_{\mathbf{i}} \mathbf{C}_{\mathbf{R,i,2}} \end{cases} . \tag{48}$$

In the second member of this vector equations all the forces are condensed in the summation.
As for the stator, it’s possible to obtain a single vector equation.
$\sum_{\mathbf{i}} \mathbf{C}_{\mathbf{R,i,2}}$ is the summation of pure moments and moments caused by forces applied in the points P_i of the rotor and calculated with reference to the vector base of system $\boldsymbol{S_2}$. If we call $\mathbf{N}_{\mathbf{R,i}}$ the pure moments in the points P_i of the rotor in $\boldsymbol{S_2}$, using the shift matrix to shift the forces from the points P_i to the center of gravity R of the rotor, we obtain

$$\begin{vmatrix} m_R\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_{\mathbf{R}} \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}}_{\mathbf{R}} \\ \dot{\boldsymbol{\omega}}_{\tau} + \dot{\boldsymbol{\omega}}_{r} \end{vmatrix} + \begin{vmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Gamma}\mathbf{J}_{\mathbf{R}} \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}}_{\mathbf{R}} \\ \boldsymbol{\omega}_{\tau} + \boldsymbol{\omega}_{r} \end{vmatrix} = \sum_{\mathbf{i}} \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{A}_{\mathbf{i,R}} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{F}_{\mathbf{R,i}} \\ \mathbf{N}_{\mathbf{R,i}} \end{vmatrix} . \tag{49}$$

In this equation, as in (27), we used the matrix $\boldsymbol{\Gamma}$ that defines the angular velocity. From now, we will make our analysis in the reference frame $\boldsymbol{S_2}$ only. Therefore we will neglect the pedice 2 in the quantities of the equations.
Before to write down the condensed form of the equations for the rotor, we have to introduce the gyroscopic matrix $\mathbf{g}_{\mathbf{R}}$.
Let’s consider the term $\boldsymbol{\omega}_{\tau} \times \mathbf{J}_{\mathbf{R}}\left(\boldsymbol{\omega}_{\tau} + \boldsymbol{\omega}_{r}\right)$ in the equation (40) and calculate it; we obtain

$$\boldsymbol{\omega}_{\tau} \times \mathbf{J}_{\mathbf{R}}\left(\boldsymbol{\omega}_{\tau} + \boldsymbol{\omega}_{r}\right) = \boldsymbol{\omega}_{\tau} \times \mathbf{J}_{\mathbf{R}}\boldsymbol{\omega}_{\tau} + \boldsymbol{\omega}_{\tau} \times \mathbf{J}_{\mathbf{R}}\boldsymbol{\omega}_{r} \,. \tag{50}$$

If $\boldsymbol{\omega}_{\tau} = \begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix}$ and $\boldsymbol{\omega}_{r} = \begin{vmatrix} \omega_R \\ 0 \\ 0 \end{vmatrix}$, always considered in the stereo-nodal reference system, we can define the last quantity in the second member of equation (50) as

$$\boldsymbol{\omega}_\tau \times \mathbf{J}_\mathbf{R}\boldsymbol{\omega}_\mathbf{r} = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix} \begin{vmatrix} J_p & 0 & 0 \\ 0 & J_d & 0 \\ 0 & 0 & J_d \end{vmatrix} \begin{vmatrix} \omega_R \\ 0 \\ 0 \end{vmatrix}, \tag{51}$$

$$\boldsymbol{\omega}_\tau \times \mathbf{J}_\mathbf{R}\boldsymbol{\omega}_\mathbf{r} = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & J_p\omega_R \\ 0 & -J_p\omega_R & 0 \end{vmatrix} \begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix} = \mathbf{g}_\mathbf{R}\boldsymbol{\omega}_\tau . \tag{52}$$

The gyroscopic matrix $\mathbf{g}_\mathbf{R}$ takes into account the gyroscopic effects due to the rotation of the rotor when we write its equilibrium equation in the stereo-nodal reference system fixed to the stator. By introducing the gyroscopic matrix in the equation (50) we obtain

$$\boldsymbol{\omega}_\tau \times \mathbf{J}_\mathbf{R}\left(\boldsymbol{\omega}_\tau + \boldsymbol{\omega}_\mathbf{r}\right) = \boldsymbol{\omega}_\tau \times \mathbf{J}_\mathbf{R}\boldsymbol{\omega}_\tau + \mathbf{g}_\mathbf{R}\boldsymbol{\omega}_\tau$$

and by using this formulation, the global equation (49), for the dynamics of the rotor, becomes

$$\begin{vmatrix} m_R\mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{J}_\mathbf{R} \end{vmatrix} \begin{vmatrix} \ddot{\mathbf{u}}_\mathbf{R} \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \begin{vmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \boldsymbol{\Gamma}\mathbf{J}_\mathbf{R} \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}}_\mathbf{R} \\ \boldsymbol{\omega}_\tau \end{vmatrix} + \begin{vmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{g}_\mathbf{R} \end{vmatrix} \begin{vmatrix} \dot{\mathbf{u}}_\mathbf{R} \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_\mathbf{i} \begin{vmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{A}_\mathbf{i,R} & \mathbf{I} \end{vmatrix} \begin{vmatrix} \mathbf{F}_\mathbf{R,i} \\ \mathbf{N}_\mathbf{R,i} \end{vmatrix} - \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_\mathbf{R}\dot{\boldsymbol{\omega}}_\mathbf{r} \end{vmatrix} \tag{53}$$

and, as already done for the stator, we can write down the following condensed vector equation for the global behavior of the rotor:

$$\mathbf{M}_\mathbf{R} \begin{vmatrix} \ddot{\mathbf{u}}_\mathbf{R} \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_\mathbf{R} \begin{vmatrix} \dot{\mathbf{u}}_\mathbf{R} \\ \boldsymbol{\omega}_\tau \end{vmatrix} + \mathbf{G}_\mathbf{R} \begin{vmatrix} \dot{\mathbf{u}}_\mathbf{R} \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_\mathbf{i} \mathbf{B}_\mathbf{R,i}\mathbf{T}_\mathbf{R,i} - \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_\mathbf{R}\dot{\boldsymbol{\omega}}_\mathbf{r} \end{vmatrix} . \tag{54}$$

The global equation for the system

To assemble the equations of stator and rotors, we have to express the displacements of the center of gravity of the rotors by using the displacements of the center of gravity of the stator. This is fundamental to have a single vector of three unknown displacements and three rotations. After this, by explicating the forces exchanged among the three rigid bodies, the three vector equations will be combined together.
Let the pedice S, 1, 2, define the stator, the first and the second rotor. Let the first rotor be a driven rotor and the second be an electric motor. The general equations for these three systems are

$$\mathbf{M}_S \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_S \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_i \mathbf{B}_{S,i}\mathbf{T}_{S,i} \ , \tag{55}$$

$$\mathbf{M}_1 \begin{vmatrix} \ddot{\mathbf{u}}_1 \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_1 \begin{vmatrix} \dot{\mathbf{u}}_1 \\ \boldsymbol{\omega}_\tau \end{vmatrix} + \mathbf{G}_1 \begin{vmatrix} \dot{\mathbf{u}}_1 \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i} - \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_1\dot{\boldsymbol{\omega}}_1 \end{vmatrix}, \tag{56}$$

$$\mathbf{M}_2 \begin{vmatrix} \ddot{\mathbf{u}}_2 \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_2 \begin{vmatrix} \dot{\mathbf{u}}_2 \\ \boldsymbol{\omega}_\tau \end{vmatrix} + \mathbf{G}_2 \begin{vmatrix} \dot{\mathbf{u}}_2 \\ \boldsymbol{\omega}_\tau \end{vmatrix} = \sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i} - \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_2\dot{\boldsymbol{\omega}}_2 \end{vmatrix}. \tag{57}$$

It’s possible to write down the following relations among the displacements of the different bodies in the inertial reference system:

$$\begin{aligned} \mathbf{u}_{1,1} &= \mathbf{u}_{S,1} + \mathbf{R}_{12}(P_{R1} - P_S)_2 \\ \dot{\mathbf{u}}_{1,1} &= \dot{\mathbf{u}}_{S,1} + \boldsymbol{\omega}_{\tau,1} \times \mathbf{R}_{12}(P_{R1} - P_S)_2 \\ \ddot{\mathbf{u}}_{1,1} &= \ddot{\mathbf{u}}_{S,1} + \dot{\boldsymbol{\omega}}_{\tau,1} \times \mathbf{R}_{12}(P_{R1} - P_S)_2 + \boldsymbol{\omega}_{\tau,1} \times \left[\boldsymbol{\omega}_{\tau,1} \times \mathbf{R}_{12}(P_{R1} - P_S)_2\right] \end{aligned} \tag{58}$$

and in the stereo-nodal reference system fixed to the stator we have

$$\begin{aligned} \mathbf{u}_{1,2} &= \mathbf{u}_{S,2} + (P_{R1} - P_S)_2 \\ \dot{\mathbf{u}}_{1,2} &= \dot{\mathbf{u}}_{S,2} + \boldsymbol{\omega}_{\tau,2} \times (P_{R1} - P_S)_2 \\ \ddot{\mathbf{u}}_{1,2} &= \ddot{\mathbf{u}}_{S,2} + \dot{\boldsymbol{\omega}}_{\tau,2} \times (P_{R1} - P_S)_2 + \boldsymbol{\omega}_{\tau,2} \times \left[\boldsymbol{\omega}_{\tau,2} \times (P_{R1} - P_S)_2\right] \end{aligned} . \tag{59}$$

Equations (59) describe, in the stereo-nodal reference system, displacement, velocity and acceleration of the rotor number 1 by means of the same quantities of the stator. It’s possible to write equations (59) in a more compact form neglecting the pedice 2 – indicating, as second index in the pedice, the stereo-nodal reference – by using the shift matrix. In fact, if $\mathbf{A}_{S,1}$ is the matrix that computes the product vectors written in the second and third equations of (59), by using the equation (8), for the rotor number 1 we have

$$\begin{vmatrix}\dot{\mathbf{u}}_1\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \begin{vmatrix}\mathbf{I} & \mathbf{A}_{S,1}\\ \mathbf{0} & \mathbf{I}\end{vmatrix}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \mathbf{B}_{S,1}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix}, \tag{60}$$

$$\begin{vmatrix}\ddot{\mathbf{u}}_1\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} = \begin{vmatrix}\mathbf{I} & \mathbf{A}_{S,1}\\ \mathbf{0} & \mathbf{I}\end{vmatrix}\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \begin{vmatrix}\mathbf{0} & \boldsymbol{\Gamma}\mathbf{A}_{S,1}\\ \mathbf{0} & \mathbf{0}\end{vmatrix}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \mathbf{B}_{S,1}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \boldsymbol{\Psi}_{S,1}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix}, \tag{61}$$

and for the rotor number 2 we have

$$\begin{vmatrix}\dot{\mathbf{u}}_2\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \begin{vmatrix}\mathbf{I} & \mathbf{A}_{S,2}\\ \mathbf{0} & \mathbf{I}\end{vmatrix}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \mathbf{B}_{S,2}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix}, \tag{62}$$

$$\begin{vmatrix}\ddot{\mathbf{u}}_2\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} = \begin{vmatrix}\mathbf{I} & \mathbf{A}_{S,2}\\ \mathbf{0} & \mathbf{I}\end{vmatrix}\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \begin{vmatrix}\mathbf{0} & \boldsymbol{\Gamma}\mathbf{A}_{S,2}\\ \mathbf{0} & \mathbf{0}\end{vmatrix}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \mathbf{B}_{S,2}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \boldsymbol{\Psi}_{S,2}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix}, \tag{63}$$

where

$$\boldsymbol{\Psi}_{S,2} = \begin{vmatrix}\mathbf{0} & \boldsymbol{\Gamma}\mathbf{A}_{S,2}\\ \mathbf{0} & \mathbf{0}\end{vmatrix}. \tag{64}$$

For the forces, it’s necessary to separate from the summation at the second member of each equation (55), (56) and (57), external forces, forces in the bearings, magnetic forces due to the motor and resistant forces due to the user. The forces coming from the bearings have to be shifted in the center of gravity of the bodies by the shift matrix as any other force. For the stator we can recognize the following forces: forces in the suspensions, forces in the bearings of the first and second rotor, magnetic torque from the electric motor, friction torque from the two rotors.
For the motor we have external forces due to the transmission and to unbalances, torque due to the magnetic field and forces in its bearings.
For the user rotor: external forces due to the transmission and to the unbalances, friction torque, resistant torque, forces in its bearings.
We have respectively for stator, rotor and motor

$$\sum_i \mathbf{B}_{S,i}\mathbf{T}_{S,i} = \sum_i \mathbf{B}_{S,i}\mathbf{T}_{S,i}^{susp} + \mathbf{B}_{S,B1}\mathbf{T}_{B1} + \mathbf{B}_{S,B2}\mathbf{T}_{B2} - \mathbf{C}_M - \mathbf{C}_F, \tag{65}$$

$$\sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i} = \sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i}^{ext} - \mathbf{B}_{1,B1}\mathbf{T}_{B1} + \mathbf{C}_F + \mathbf{C}_U, \tag{66}$$

$$\sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i} = \sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i}^{ext} - \mathbf{B}_{2,B2}\mathbf{T}_{B2} + \mathbf{C}_M. \tag{67}$$

Let’s replace the displacements of the rotors with relations (60) ÷ (63). The equations for the rotors become

$$\mathbf{M}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \mathbf{M}_1\boldsymbol{\Psi}_{S,1}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \mathbf{G}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i} - \begin{vmatrix}\mathbf{0}\\ \mathbf{J}_1\dot{\boldsymbol{\omega}}_1\end{vmatrix}, \quad (68)$$

$$\mathbf{M}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \mathbf{M}_2\boldsymbol{\Psi}_{S,2}\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \boldsymbol{\Lambda}\mathbf{M}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \mathbf{G}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i} - \begin{vmatrix}\mathbf{0}\\ \mathbf{J}_2\dot{\boldsymbol{\omega}}_2\end{vmatrix}. \quad (69)$$

Before to continue, it's necessary to introduce the linear form of the equations with reference to the hypothesis of small rotations. In fact the second factor in the equation of the stator and the second and the third in the equations of the rotors are negligible for this reason (see appendix 1). Therefore we can write the new linearized equations with the description of the forces as

$$\mathbf{M}_S\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{S,i}\mathbf{T}_{S,i}^{susp} + \mathbf{B}_{S,B1}\mathbf{T}_{B1} + \mathbf{B}_{S,B2}\mathbf{T}_{B2} - \mathbf{C}_M - \mathbf{C}_F, \quad (70)$$

$$\mathbf{M}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \mathbf{G}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i}^{ext} - \mathbf{B}_{1,B1}\mathbf{T}_{B1} + \mathbf{C}_F + \mathbf{C}_U - \begin{vmatrix}\mathbf{0}\\ \mathbf{J}_1\dot{\boldsymbol{\omega}}_1\end{vmatrix}, \quad (71)$$

$$\mathbf{M}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \mathbf{G}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i}^{ext} - \mathbf{B}_{2,B2}\mathbf{T}_{B2} + \mathbf{C}_M - \begin{vmatrix}\mathbf{0}\\ \mathbf{J}_2\dot{\boldsymbol{\omega}}_2\end{vmatrix}. \quad (72)$$

By using equation (71) and equation (72), it's possible to derive forces in the bearings and replace them in the equation (72). These forces are, respectively for the driven rotor and for the motor,

$$\mathbf{T}_{B1} = \mathbf{B}_{B1,1}\sum_i \mathbf{B}_{1,i}\mathbf{T}_{1,i}^{ext} - \mathbf{B}_{B1,1}\mathbf{M}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} - \mathbf{B}_{B1,1}\mathbf{G}_1\mathbf{B}_{S,1}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \mathbf{B}_{B1,1}(\mathbf{C}_F + \mathbf{C}_U) - \mathbf{B}_{B1,1}\begin{vmatrix}\mathbf{0}\\ \mathbf{J}_1\dot{\boldsymbol{\omega}}_1\end{vmatrix}, \quad (73)$$

$$\mathbf{T}_{B2} = \mathbf{B}_{B2,2}\sum_i \mathbf{B}_{2,i}\mathbf{T}_{2,i}^{ext} - \mathbf{B}_{B2,2}\mathbf{M}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\ddot{\mathbf{u}}_S\\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} - \mathbf{B}_{B2,2}\mathbf{G}_2\mathbf{B}_{S,2}^T\begin{vmatrix}\dot{\mathbf{u}}_S\\ \boldsymbol{\omega}_\tau\end{vmatrix} + \mathbf{B}_{B2,2}\mathbf{C}_M - \mathbf{B}_{B2,2}\begin{vmatrix}\mathbf{0}\\ \mathbf{J}_2\dot{\boldsymbol{\omega}}_2\end{vmatrix}. \quad (74)$$

By replacing this forces in the equation (70) we obtain the global equation for the system:

$$\mathbf{M}_S \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} = \sum_i \mathbf{B}_{S,i}\mathbf{T}_{S,i}^{susp} + \sum_i \mathbf{B}_{S,i}\mathbf{T}_{1,i}^{ext} - \mathbf{B}_{S,1}\mathbf{M}_1\mathbf{B}_{S,1}^T \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} - \mathbf{B}_{S,1}\mathbf{G}_1\mathbf{B}_{S,1}^T \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} + \mathbf{C}_U -$$

$$- \mathbf{B}_{S,1} \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_1\dot{\boldsymbol{\varphi}}_1 \end{vmatrix} + \sum_i \mathbf{B}_{S,i}\mathbf{T}_{2,i}^{ext} - \mathbf{B}_{S,2}\mathbf{M}_2\mathbf{B}_{S,2}^T \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} - \mathbf{B}_{S,2}\mathbf{G}_2\mathbf{B}_{S,2}^T \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} - \mathbf{B}_{S,2} \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_2\dot{\boldsymbol{\varphi}}_2 \end{vmatrix} , \quad (75)$$

By reordering this equation we have

$$\left(\mathbf{M}_S + \mathbf{B}_{S,1}\mathbf{M}_1\mathbf{B}_{S,1}^T + \mathbf{B}_{S,2}\mathbf{M}_2\mathbf{B}_{S,2}^T \right) \begin{vmatrix} \ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau \end{vmatrix} + \left(\mathbf{B}_{S,1}\mathbf{G}_1\mathbf{B}_{S,1}^T + \mathbf{B}_{S,2}\mathbf{G}_2\mathbf{B}_{S,2}^T \right) \begin{vmatrix} \dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau \end{vmatrix} =$$

$$= \sum_i \mathbf{B}_{S,i} \left(\mathbf{T}_{S,i}^{susp} + \mathbf{T}_{1,i}^{ext} + \mathbf{T}_{2,i}^{ext} \right) + \mathbf{C}_U - \mathbf{B}_{S,1} \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_1\dot{\boldsymbol{\varphi}}_1 \end{vmatrix} - \mathbf{B}_{S,2} \begin{vmatrix} \mathbf{0} \\ \mathbf{J}_2\dot{\boldsymbol{\varphi}}_2 \end{vmatrix} , \quad (76)$$

If we analyze the first member of the equation we can recognize, in the first quantity, the global mass matrix of the system and, in the second, the global gyroscopic matrix. At the second member of the equation, the first summation is the resultant generalized force acting on the system and the other quantities are the effects of the rotor inertia referred to the rotation axes in the case the rotor is accelerating. The equation (76) is similar to the equation of a single rigid body with more inertial effects of the rotors and, fundamental, the gyroscopic effects due to the rotation.

MULTI-ROTOR SYSTEMS

If we consider a multi-rotor system, it's possible to use the previous global equation (76) for two rotors and obtain a general expression to describe, very simply, the dynamics of a so complex system. By generalizing equation (76) we obtain

$$\left(\mathbf{M}_S + \mathbf{B}_{S,j}\mathbf{M}_j\mathbf{B}_{S,j}^T\right)\begin{vmatrix}\ddot{\mathbf{u}}_S \\ \dot{\boldsymbol{\omega}}_\tau\end{vmatrix} + \mathbf{B}_{S,j}\mathbf{G}_j\mathbf{B}_{S,j}^T\begin{vmatrix}\dot{\mathbf{u}}_S \\ \boldsymbol{\omega}_\tau\end{vmatrix} = \sum_i \mathbf{B}_{S,i}\left(\mathbf{T}_{S,i}^{susp} + \mathbf{T}_{j,i}^{ext}\right) + \mathbf{C}_U - \mathbf{B}_{S,j}\begin{vmatrix}\mathbf{0} \\ \mathbf{J}_j\dot{\boldsymbol{\omega}}_j\end{vmatrix}. \tag{77}$$

As said before, it's possible to recognize a global mass matrix, a global gyroscopic matrix and the summation of the forces and the inertial effects. The global mass matrix is the sum of the mass matrixes of the different bodies with reference to the center of gravity of the stator. That means the transformation of the mass matrixes of the rotors, by the shift matrix, takes into account the center of gravity of the rotors could be different of the center of the stator. The global gyroscopic matrix is the sum of the gyroscopic effects of the rotors reported to the center of gravity of the stator by the same transformation. The last quantity in the equation (77) is the sum of the inertial effects of the rotors when they are accelerating.

It's always possible to consider complex mechanical systems when we are able to describe interaction forces among components, and to build a global model by describing forces in the reference system of the main body, used to describe the motion. For example it's possible to consider the dynamics of an engine when forces between piston, crank shaft and basement are known. This because the principles we use to find out equations of motion, are the Newton's laws.