RELATIVE KINEMATICS

The equations of motion for a point *P* will be analyzed in two different reference systems. One reference system is inertial, fixed to the ground, the second system is moving in the physical space and the point *P* is also free to move in any configuration of the space (figure 1).
For the moment, *P* is considered without mass, and only the kinematic aspect will be analyzed. This helps to introduce the very important concept of the angular velocity vector.
If the point *P* is moving in the reference system $\boldsymbol{S_1}$, $\mathbf{u}_1(t)$ is the vector that describes its position at the time *t* in this system. *P* is also moving respect to the mobile reference system $\boldsymbol{S_2}$ and $\mathbf{q}_2(t)$ is the vector that describes its position in this system.
Now, it's possible to express the position of *P* by a vector equation where the relation between the two systems appears. In fact it's

$$\mathbf{u}_1(t) = \mathbf{r}_1(t) + \mathbf{R}_{12}(t)\mathbf{q}_2(t), \tag{1}$$

where $\mathbf{r}_1(t)$ is the position of the center O_2 in the system $\boldsymbol{S_1}$ and $\mathbf{R}_{12}(t)$ is the rotation matrix that translates a vector from the vector base of $\boldsymbol{S_2}$ to the base of $\boldsymbol{S_1}$.

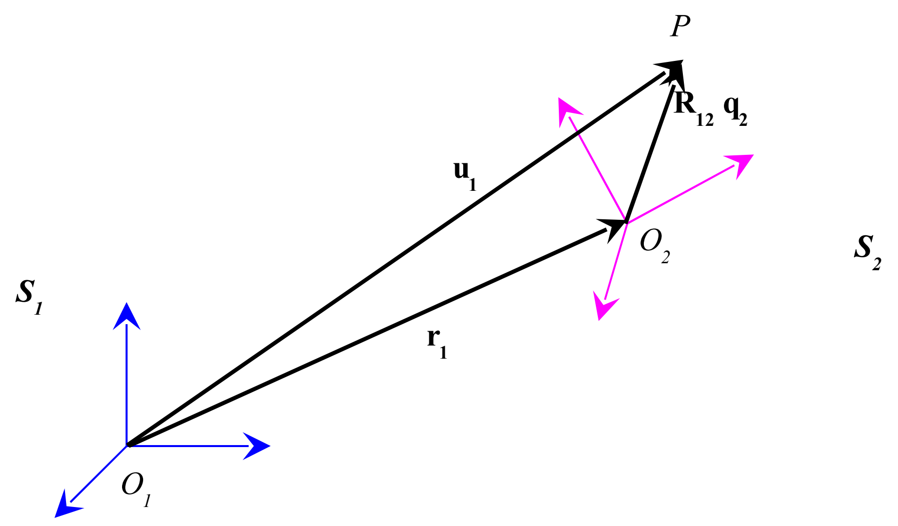

Figure 1

Before to express velocity and acceleration of the point *P*, it's necessary to consider some important properties of the rotation matrix.
To give a more simple formulation, in the following sections time dependence will be not always explicitly indicated.

Properties of the rotation matrix

For the property of the rotation matrix to be orthogonal, it is

$$\mathbf{R}_{12}\mathbf{R}_{12}^{\mathbf{T}} = \mathbf{I}\,. \qquad (2)$$

When this equation is derived with respect to time we obtain

$$\frac{d}{dt}\left(\mathbf{R}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right) = \dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} + \mathbf{R}_{12}\dot{\mathbf{R}}_{12}^{\mathbf{T}} = \mathbf{0}\,. \qquad (3)$$

Now, it's possible to define a skew symmetric matrix **Γ** with the following property:

$$\mathbf{\Gamma}^{\mathbf{T}} = \left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right)^{\mathbf{T}} = \mathbf{R}_{12}\dot{\mathbf{R}}_{12}^{\mathbf{T}} = -\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} = -\mathbf{\Gamma}\,; \qquad (4)$$

and the components of this matrix are

$$\mathbf{\Gamma} = \dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix}. \qquad (5)$$

The matrix **Γ** defines the angular velocity of the system $\boldsymbol{S_2}$ with reference to the system $\boldsymbol{S_1}$. The product of the matrix **Γ** with a vector **v** is equal to the vector product between a new vector **ω** and the vector **v**. In fact it's possible to express this result as

$$\mathbf{\Gamma v} = \boldsymbol{\omega} \times \mathbf{v}\,. \qquad (6)$$

ω is the angular velocity vector and

$$\boldsymbol{\omega} = \begin{vmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{vmatrix}. \qquad (7)$$

For the second derivative of the equation (2) we obtain by simple operations

$$\frac{d}{dt}\left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right) = \ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} + \dot{\mathbf{R}}_{12}\dot{\mathbf{R}}_{12}^{\mathbf{T}} = \ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} + \left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right)\left(\mathbf{R}_{12}\dot{\mathbf{R}}_{12}^{\mathbf{T}}\right). \tag{8}$$

It's important to explicate, from the (8), the term $\ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}$, that it will let us to calculate the expression for the acceleration of the point *P* in the global reference system. This term is

$$\ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} = \frac{d}{dt}\left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right) - \left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right)\left(\mathbf{R}_{12}\dot{\mathbf{R}}_{12}^{\mathbf{T}}\right) \tag{9}$$

and for the equation (5) it's possible to write

$$\ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}} = \frac{d}{dt}\left(\boldsymbol{\Gamma}\right) + \boldsymbol{\Gamma}^{2} \tag{10}$$

and for the same argument of the (6), if **v** is a vector defined in the global reference system, we have the following vector operation:

$$\left(\ddot{\mathbf{R}}_{12}\mathbf{R}_{12}^{\mathbf{T}}\right)\mathbf{v} = \left[\frac{d}{dt}\left(\boldsymbol{\Gamma}\right) + \boldsymbol{\Gamma}^{2}\right]\mathbf{v} = \dot{\boldsymbol{\omega}}\times\mathbf{v} + \boldsymbol{\omega}\times\left(\boldsymbol{\omega}\times\mathbf{v}\right). \tag{11}$$

Velocity and acceleration

By differentiating the fundamental equation (1) with respect to time, we calculate velocity and acceleration of the point *P*. For the velocity of the point in the inertial reference system we have

$$\dot{\mathbf{u}}_1(t) = \left[\dot{\mathbf{r}}_1(t) + \dot{\mathbf{R}}_{12}(t)\,\mathbf{q}_2(t)\right] + \mathbf{R}_{12}(t)\,\dot{\mathbf{q}}_2(t)\,. \tag{12}$$

It's more interesting to write this equation using the following expression where three velocity vectors compare:

$$\mathbf{v}_\mathbf{a} = \mathbf{v}_\tau + \mathbf{v}_\mathbf{r}\,. \tag{13}$$

$\mathbf{v}_\mathbf{a}$ is the absolute velocity, i.e., the velocity of the point *P* measured with reference to the global reference system and it corresponds to the vector

$$\mathbf{v}_\mathbf{a} = \dot{\mathbf{u}}_1(t)\,; \tag{14}$$

$\mathbf{v}_\mathbf{r}$ is the relative velocity of the point *P*, i.e. the velocity of the point measured in the mobile reference system, but expressed using the versors of the global reference. In fact, using the rotation matrix between the two reference systems, we have

$$\mathbf{v}_\mathbf{r} = \mathbf{R}_{12}(t)\,\dot{\mathbf{q}}_2(t)\,; \tag{15}$$

$\mathbf{v}_\tau$ is the drag velocity, that is the velocity of the point P as if it was fixed to the mobile reference system and moving with it:

$$\mathbf{v}_\tau = \dot{\mathbf{r}}_1(t) + \dot{\mathbf{R}}_{12}(t)\,\mathbf{q}_2(t)\,. \tag{16}$$

It's interesting to write down the drag velocity as

$$\mathbf{v}_\tau = \mathbf{v}_{\mathbf{O}_2} + \boldsymbol{\omega} \times (P - O_2)\,. \tag{17}$$

Such formula expresses the motion law for a rigid body when it's fixed with the mobile reference system. For the equations (1) and (12) we obtain

$$\mathbf{v}_\tau = \dot{\mathbf{r}}_1 + \dot{\mathbf{R}}_{12}\mathbf{q}_2 = \dot{\mathbf{r}}_1 + \dot{\mathbf{R}}_{12}\mathbf{R}_{12}{}^{\mathrm{T}}(\mathbf{u}_1 - \mathbf{r}_1) = \dot{\mathbf{r}}_1 + \boldsymbol{\omega}\times(\mathbf{u}_1 - \mathbf{r}_1). \qquad (18)$$

Differentiating the equation (12) with respect to time, for the acceleration of the point *P*, we have

$$\ddot{\mathbf{u}}_1 = \left(\ddot{\mathbf{r}}_1 + \ddot{\mathbf{R}}_{12}\mathbf{q}_2\right) + 2\dot{\mathbf{R}}_{12}\dot{\mathbf{q}}_2 + \mathbf{R}_{12}\ddot{\mathbf{q}}_2. \qquad (19)$$

We can write this equation as a sum of three accelerations:

$$\mathbf{a}_\mathbf{a} = \mathbf{a}_\tau + \mathbf{a}_\mathbf{c} + \mathbf{a}_\mathbf{r}. \qquad (20)$$

$\mathbf{a}_\mathbf{a}$ is the absolute acceleration, i.e., the acceleration of the point *P* measured with reference to the global reference system and it corresponds to the vector

$$\mathbf{a}_\mathbf{a} = \ddot{\mathbf{u}}_1(t); \qquad (21)$$

$\mathbf{a}_\tau$ is the drag acceleration, and for the equation (10) it's possible to write down

$$\mathbf{a}_\tau = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{R}}_{12}\mathbf{q}_2 = \ddot{\mathbf{r}}_1 + \ddot{\mathbf{R}}_{12}\mathbf{R}_{12}{}^{\mathrm{T}}(\mathbf{u}_1 - \mathbf{r}_1) = \ddot{\mathbf{r}}_1 + \dot{\boldsymbol{\Gamma}}(\mathbf{u}_1 - \mathbf{r}_1) + \boldsymbol{\Gamma}^2(\mathbf{u}_1 - \mathbf{r}_1), \qquad (22)$$

$$\mathbf{a}_\tau = \ddot{\mathbf{r}}_1 + \dot{\boldsymbol{\omega}}\times(\mathbf{u}_1 - \mathbf{r}_1) + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(\mathbf{u}_1 - \mathbf{r}_1)], \qquad (23)$$

$$\mathbf{a}_\tau = \mathbf{a}_{\mathbf{O}_2} + \dot{\boldsymbol{\omega}}\times(P - O_2) + \boldsymbol{\omega}\times[\boldsymbol{\omega}\times(P - O_2)]; \qquad (24)$$

$\mathbf{a}_\mathbf{r}$ is the relative acceleration, i.e., the acceleration of the point P referred to the mobile reference system but expressed by the versors of the inertial:

$$\mathbf{a}_\mathbf{r} = \mathbf{R}_{12}\ddot{\mathbf{q}}_2; \qquad (25)$$

$\mathbf{a}_\mathbf{c}$ is the Coriolis acceleration defined in the global reference system as

$$\mathbf{a}_\mathbf{c} = 2\dot{\mathbf{R}}_{12}\dot{\mathbf{q}}_2 = 2\left(\dot{\mathbf{R}}_{12}\mathbf{R}_{12}{}^{\mathrm{T}}\right)\left(\mathbf{R}_{12}\dot{\mathbf{q}}_2\right) = 2\boldsymbol{\Gamma}\mathbf{v}_\mathbf{r} = 2\boldsymbol{\omega}\times\mathbf{v}_\mathbf{r}. \qquad (26)$$