THE ANGULAR VELOCITY VECTOR

Let $\{\Xi, \varepsilon_1, \varepsilon_2, \varepsilon_3\}$ and $\{O, e_1, e_2, e_3\}$ be the inertial reference system and the mobile reference system fixed to the rigid body.
The angular velocity vector describing the motion of the mobile reference system with reference to the inertial is

$$\boldsymbol{\omega} = \left(\frac{d\mathbf{e}_2}{dt}\cdot\mathbf{e}_3\right)\mathbf{e}_1 + \left(\frac{d\mathbf{e}_3}{dt}\cdot\mathbf{e}_1\right)\mathbf{e}_2 + \left(\frac{d\mathbf{e}_1}{dt}\cdot\mathbf{e}_2\right)\mathbf{e}_3 . \tag{1}$$

If we consider a general vector **v** fixed with the mobile reference system, its derivative with respect to time will be

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\omega}\times\mathbf{v}, \tag{2}$$

and in a matrix formulation we can write

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{\Omega}\mathbf{v}, \tag{3}$$

where

$$\boldsymbol{\Omega} = \begin{vmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{vmatrix}. \tag{4)}$$

Now it’s possible to express the components of the angular velocity as

$$\omega_1 = \frac{d\mathbf{e}_2}{dt}\cdot\mathbf{e}_3 = \frac{d}{dt}\left[(\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right]\cdot\left[(\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right], \tag{5}$$

$$\omega_2 = \frac{d\mathbf{e}_3}{dt}\cdot\mathbf{e}_1 = \frac{d}{dt}\left[(\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_3\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right]\cdot\left[(\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right], \tag{6}$$

$$\omega_3 = \frac{d\mathbf{e}_1}{dt}\cdot\mathbf{e}_2 = \frac{d}{dt}\left[(\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_1\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right]\cdot\left[(\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_1)\boldsymbol{\varepsilon}_1 + (\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_2)\boldsymbol{\varepsilon}_2 + (\mathbf{e}_2\cdot\boldsymbol{\varepsilon}_3)\boldsymbol{\varepsilon}_3\right]. \tag{7}$$

The matrix of rotation from the mobile reference system to the inertial is

$$\mathbf{R} = \begin{vmatrix} \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_1 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_2 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_3 \end{vmatrix} \tag{8}$$

therefore, the angular velocity results

$$\boldsymbol{\Omega} = \frac{d}{dt}\left(\begin{vmatrix} \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_1 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_2 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_3 \end{vmatrix} \right) \begin{vmatrix} \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_1 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_1 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_2 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_2 \\ \mathbf{e}_1 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_2 \cdot \boldsymbol{\varepsilon}_3 & \mathbf{e}_3 \cdot \boldsymbol{\varepsilon}_3 \end{vmatrix}^{\mathbf{T}}, \tag{9}$$

$$\boldsymbol{\Omega} = \dot{\mathbf{R}}\mathbf{R}^{\mathbf{T}}.$$